1) $\vec{F} = \nabla f \Rightarrow \langle \sin y, x\cos y + \cos z, -y\sin z \rangle = \langle f_x, f_y, f_z \rangle$

$\Rightarrow f_x(x,y,z) = \sin y$, $f_y(x,y,z) = x\cos y$, $f_z(x,y,z) = -y\sin z$

$f_x(x,y,z) = \sin y \Rightarrow f(x,y,z) = x\sin y + g(y,z)$.

Next, $f_y(x,y,z) = x\cos y + \cos z \Rightarrow x\cos y + g_y(y,z) = x\cos y + \cos z \Rightarrow g_y(y,z) = \cos z$
$\Rightarrow g(y) = y\cos z + h(z)$.

$f_z(x,y,z) = -y\sin z \Rightarrow \frac{\partial}{\partial z}\left(x\sin(y) + y\cos z + h(z)\right) = -y\sin z \Rightarrow -y\sin z + h'(z) = -y\sin z \Rightarrow$

$\Rightarrow h'(z) = 0 \Rightarrow h(z) = K$

Therefore, $f(x,y,z) = x\sin y + y\cos z + K$ and taking $K=0$, we have $f(x,y,z) = x\sin y + y\cos z$

$$\text{b)} \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{F.T of line integrals}}{=} \underbrace{f(\vec{r}(\pi/2))}_{\text{terminal pt}} - \underbrace{f(\vec{r}(0))}_{\text{initial pt}} = f(1, \pi/2, \pi) - f(0,0,0) = 1 - \frac{\pi}{2}.$$

2) The functions $P(x,y) = 2xe^{-y}$ and $Q(x,y) = 2y - x^2e^{-y}$
have continuous first order derivatives on all of $\mathbb{R}^2$ (which is open
and simply connected).

Then $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y}$ & $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y - x^2e^{-y}) = -2xe^{-y}$

Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on a simple connected and open region $\mathbb{R}^2$ and P and Q have continuous first order derivatives, and thus $\vec{F}$ is conservative.

As $\vec{F}$ is conservative, $\int_C \vec{F}\cdot d\vec{r}$ is path independent and also a potential function exists i.e.

we can find a function $f(x,y)$ such that $\nabla f = \vec{F}$

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Then $f_x(x,y) = 2xe^{-y} \Rightarrow f(x,y) = x^2e^{-y} + g(y)$

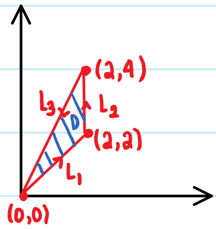
Now,

$f_y = -x^2e^{-y} + g'(y) \Rightarrow 2y - x^2e^{-y} = -x^2e^{-y} + g'(y) \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + K$

Therefore, $f(x,y) = x^2e^{-y} + y^2$ (taking $K=0$).

Then, $\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - 1 = \frac{4}{e}$.

3)



$L_1 : y = x$, $L_2 : x = 2$, $L_3 : y = 2x$

The region D enclosed by the curve C is given by $D = \{(x,y) \mid 0 \le x \le 2,\ x \le y \le 2x\}$, so

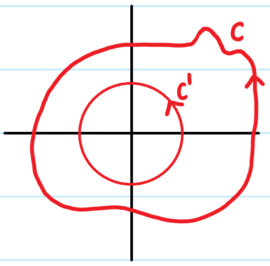
by Green's Thm,

$$\int_C xy^2dx + 2xy\,dy = \iint_D \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(xy^2)\,dA = \int_0^2\int_x^{2x} 2y - 2xy\,dy\,dx = \int_0^2 \Big[y^2 - xy^2\Big]_x^{2x} dy\,dx$$

$$= \int_0^2 4x^2 - 4x^3 - x^2 + x^3\,dx = \int_0^2 3x^2 - 3x^3 = x^3 - \frac{3x^4}{4}\Big]_0^2 = 8 - 12 = -4$$

4) Let $P(x,y) = \frac{2xy}{(x^2+y^2)^2}$, $Q(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$

Let C be an arbitrary closed path that encloses the origin. Let C' be a counterclockwise-oriented circle w/ center the origin and radius a, where a is chosen to be small enough so that C' lies C, and D be the region bounded by C and C'. Then it's positively oriented boundary is $C \cup (-C')$.



$$\frac{\partial P}{\partial y} = \frac{2x(x^2+y^2)^2 - 2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

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$$\frac{\partial Q}{\partial x} = \frac{-2x(x^2+y^2) - (y^2-x^2).2(x^2+y^2).2x}{(x^2+y^2)^4} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Now the general version of Green's Theorem gives us

$$\int_{C\cup(-C')} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \Rightarrow \int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D 0\,dA \Rightarrow$$

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy \quad \text{and} \quad \int_C \vec{F}.d\vec{r} = \int_{C'} \vec{F}.d\vec{r}$$

Parametrize C' as $\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}$, $0 \leq t \leq 2\pi$.

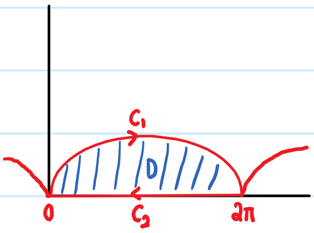
Then,

$$\int_C \vec{F}.d\vec{r} = \int_{C'} \vec{F}.d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)).\vec{r}\,'(t)\,dt = \int_0^{2\pi} \frac{2(a\cos t)(a\sin t)\hat{\imath} + (a^2\sin^2 t - a^2\cos^2 t)\hat{\jmath}}{(a^2\cos^2 t + a^2\sin^2 t)^2} \cdot \left(-a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath}\right) dt$$

$$= \frac{1}{a}\int_0^{2\pi} (-\cos t\sin^2 t - \cos^3 t)\,dt = \frac{1}{a}\int_0^{2\pi} -\cos t\,(\sin^2 t + \cos^2 t)\,dt$$

$$= \frac{1}{a}\int_0^{2\pi} \cos t\,dt = -\frac{1}{a}\Big[\sin t\Big]_0^{2\pi} = 0.$$

5)



C_1 ≡ arch of the cycloid from $(0,0)$ to $(2\pi,0)$ which corresponds to $(0,2\pi)$ given by $x(t) = t - \sin t$ and $y = 1-\cos t$.

- C_2 be the line segment $x = 2\pi - t$, $y = 0$, $0 \leq t \leq 2\pi$, and let $C = C_1 \cup C_2$.

Note that $-C$ is positively oriented and we want to find the area of the region enclosed by D.

$$A(D) = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt) = \int_0^{2\pi} 1 - 2\cos t + \cos^2 t\,dt$$

$$= \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 2\pi + \frac{1}{2}.2\pi = 3\pi.$$

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6) $\vec{F} = \left\langle \frac{x}{y}, \frac{y}{z}, \frac{z}{x} \right\rangle$

• $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{y} & \frac{y}{z} & \frac{z}{x} \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{z}{x}\right) - \frac{\partial}{\partial z}\left(\frac{y}{z}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{z}{x}\right) - \frac{\partial}{\partial z}\left(\frac{x}{y}\right)\right)$

$+ \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{y}{z}\right) - \frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right) = \hat{i}\left(0 + \frac{y}{z^2}\right) - \hat{j}\left(-\frac{z}{x^2} - 0\right) + \hat{k}\left(0 + \frac{x}{y^2}\right) = \left\langle \frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2} \right\rangle.$

• $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial y}\left(\frac{y}{z}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x}\right) = \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$

7) a) let $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$

$$\text{div}(f\vec{F}) = \text{div}(fP\hat{i} + fQ\hat{j} + fR\hat{k}) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

$$= f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial x}P + f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial y}Q + f\frac{\partial R}{\partial z} + \frac{\partial f}{\partial z}R$$

$$= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + f_x P + f_y Q + f_z R$$

$$= f\,\text{div}\vec{F} + \langle f_x, f_y, f_z \rangle \cdot \langle P, Q, R \rangle = f\,\text{div}\vec{F} + \vec{F} \cdot \nabla f$$

b) $\nabla^2 f = \text{div}(\nabla f) = \text{div}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

c) $\text{curl}(\text{curl } \vec{F}) = \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} & \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{vmatrix}$

$= \left(\frac{\partial^2 Q}{\partial y \partial x} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 R}{\partial z \partial x}\right)\hat{i} + \left(\frac{\partial^2 R}{\partial z \partial y} - \frac{\partial^2 Q}{\partial^2 z} - \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 P}{\partial x \partial y}\right)\hat{j} +$

$+ \left(\frac{\partial^2 P}{\partial x \partial z} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial x \partial y} + \frac{\partial^2 Q}{\partial y \partial z}\right)\hat{k}.$

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$$\operatorname{grad}(\operatorname{div}\vec{F}) - \nabla^2\vec{F} = \operatorname{grad}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) - \nabla^2 P\,\hat{i} + \nabla^2 Q\,\hat{j} + \nabla^2 R\,\hat{k}$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x\partial y} + \frac{\partial^2 R}{\partial y\partial z}\right)\hat{i} + \left(\frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial y\partial z}\right)\hat{j} + \left(\frac{\partial^2 P}{\partial z\partial x} + \frac{\partial^2 Q}{\partial z\partial y} + \frac{\partial^2 R}{\partial z^2}\right)\hat{j}$$

$$- \left[\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right)\hat{i} + \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}\right)\hat{j} + \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2}\right)\hat{k}\right]$$

$$= \left(\frac{\partial^2 Q}{\underline{\partial x\partial y}} + \frac{\partial^2 R}{\underline{\partial x\partial z}} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2}\right)\hat{i} + \left(\frac{\partial^2 P}{\underline{\partial y\partial x}} + \frac{\partial^2 R}{\underline{\partial y\partial z}} - \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial z^2}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 P}{\underline{\partial z\partial x}} + \frac{\partial^2 Q}{\underline{\partial x\partial y}} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2}\right)\hat{k}$$

Then applying Clairaut's Theorem to switch the order of differentiation in the mixed partial derivatives, we see that

$$\operatorname{curl}(\operatorname{curl}\vec{F}) = \operatorname{grad}(\operatorname{div}\vec{F}) - \nabla^2\vec{F}.$$

8a) Let $\vec{F} = f\,\nabla g$

Then,

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \operatorname{div}(f\,\nabla g)\,dA = \iint_D f\,\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f\,dA$$

$$= \iint_D f\,\nabla^2 g + \nabla f\cdot\nabla g\,dA = \iint_D f\,\nabla^2 g\,dA + \iint_D \nabla f\cdot\nabla g\,dA$$

$$\Rightarrow \iint_D f\,\nabla^2 g\,dA = \oint_C f\,\nabla g\cdot\hat{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

b)

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b) $$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right) dA = \iint_D f\nabla^2 g\, dA - \iint_D g\nabla^2 f\, dA$$

Now we can use answer from part a) to rewrite the RHS as

$$= \oint_C f(\nabla g)\hat{n}\, ds - \iint_D \nabla f\cdot\nabla g\, dA - \left(\oint_C g(\nabla f)\cdot\hat{n}\, ds - \iint_D \nabla g\cdot\nabla f\, dA\right)$$

$$= \oint_C f(\nabla g)\hat{n}\, ds - \iint_D \nabla f\cdot\nabla g\, dA - \oint_C g(\nabla f)\cdot\hat{n}\, ds + \iint_D \nabla f\cdot\nabla g\, dA$$

$$= \oint_C f(\nabla g)\cdot\hat{n}\, ds - \oint_C g(\nabla f)\cdot\hat{n}\, ds$$

$$= \oint_C (f\nabla g - g\nabla f)\cdot\hat{n}\, ds$$

c) Assume $\nabla^2 g = 0$ on D.

Then by Green's first identity w/ $f(x,y) = 1$,

$$\iint_D 1\underbrace{\nabla^2 g}_{0}\, dA = \oint_C 1(\nabla g)\cdot\hat{n}\, ds - \iint_D \underbrace{\nabla(1)}_{0}\cdot\nabla(g)\, dA$$

$$\Rightarrow 0 = \oint_C \underbrace{\nabla g\cdot\hat{n}}_{D_{\hat{n}}(g)}\, ds \Rightarrow \oint_C D_{\hat{n}} g\, ds = 0.$$

d) Use Green's first identity $g(x,y) = f(x,y)$.

$$\iint_D f\underbrace{\nabla^2 f}_{0}\, dA = \oint_C \underbrace{f}_{f=0 \text{ on } C}(\nabla g)\cdot\hat{n}\, ds - \iint_D \nabla f\cdot\nabla f\, dA$$

$$\Rightarrow 0 = 0 - \iint_D |\nabla f|^2 dA \Rightarrow \iint_D |\nabla f|^2 dA = 0.$$

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9) $\vec{H} = \langle h_1, h_2, h_3 \rangle$; $\vec{E} = \langle e_1, e_2, e_3 \rangle$

a) $\nabla \times (\nabla \times \vec{E}) = \nabla \times (\text{curl } \vec{E}) = \nabla \times \left(-\frac{1}{c} \frac{\partial H}{\partial t} \right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{1}{c}\frac{\partial h_1}{\partial t} & -\frac{1}{c}\frac{\partial h_2}{\partial t} & -\frac{1}{c}\frac{\partial h_3}{\partial t} \end{vmatrix} = -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y \partial t} - \frac{\partial^2 h_2}{\partial z \partial t} \right)\hat{i} + \left(\frac{\partial^2 h_1}{\partial z \partial t} - \frac{\partial^2 h_3}{\partial x \partial t} \right)\hat{j} + \left(\frac{\partial^2 h_2}{\partial x \partial t} - \frac{\partial^2 h_1}{\partial y \partial t} \right)\hat{k} \right]$$

Now assuming that all second order partial derivatives are continuous so that we can switch order of derivatives

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right)\hat{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right)\hat{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right)\hat{k} \right] = -\frac{1}{c}\frac{\partial}{\partial t}\left(\text{curl } \vec{H} \right)$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \vec{E}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} .$$

b) $\nabla \times (\nabla \times \vec{H}) = \nabla \times (\text{curl } \vec{H}) = \nabla \times \left(\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{1}{c}\frac{\partial e_1}{\partial t} & -\frac{1}{c}\frac{\partial e_2}{\partial t} & -\frac{1}{c}\frac{\partial e_3}{\partial t} \end{vmatrix} = \frac{1}{c}\left[\left(\frac{\partial^2 e_3}{\partial y \partial t} - \frac{\partial^2 e_2}{\partial z \partial t} \right)\hat{i} + \left(\frac{\partial^2 e_1}{\partial z \partial t} - \frac{\partial^2 e_3}{\partial x \partial t} \right)\hat{j} + \left(\frac{\partial^2 e_2}{\partial x \partial t} - \frac{\partial^2 e_1}{\partial y \partial t} \right)\hat{k} \right]$$

Now assuming that all second order partial derivatives are continuous so that we can switch order of derivatives

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial e_3}{\partial y} - \frac{\partial e_2}{\partial z} \right)\hat{i} + \left(\frac{\partial e_1}{\partial z} - \frac{\partial e_3}{\partial x} \right)\hat{j} + \left(\frac{\partial e_2}{\partial x} - \frac{\partial e_1}{\partial y} \right)\hat{k} \right] = \frac{1}{c}\frac{\partial}{\partial t}\left(\text{curl } \vec{E} \right)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} .$$

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c) Note from equation (1),

$$\nabla^2 \vec{E} = \text{grad}\,(\text{div}\,\vec{E}) - \text{curl}\,(\text{curl}\,\vec{E})$$

$$= \text{grad}\,(0) - \underbrace{\left(-\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}\right)}_{9a)} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

d) Note from equation (1),

$$\nabla^2 \vec{H} \quad \text{grad}\,(\text{div}\,\vec{H}) - \text{curl}\,(\text{curl}\,\vec{H})$$

$$= \text{grad}\,(0) - \underbrace{\left(-\frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2}\right)}_{9b)} = \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2}.$$

10) $\vec{F}(\vec{r}(t)) = m\vec{r}\,''(t)$.

Then,

$$W = \int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)\,dt = \int_a^b m\vec{r}\,''(t).\vec{r}\,'(t)\,dt$$

Note that $\frac{1}{2}\frac{d}{dt}\left(\vec{r}\,'(t).\vec{r}\,'(t)\right) = \frac{1}{2}\left[\vec{r}\,''(t).\vec{r}\,'(t) + \vec{r}\,'(t).\vec{r}\,''(t)\right] = \vec{r}\,''(t).\vec{r}\,'(t)$

$$\text{So, } W = \frac{1}{2}m\int_a^b \frac{d}{dt}\left[\vec{r}\,'(t).\vec{r}\,'(t)\right] dt$$

$$= \frac{1}{2}m\left[\vec{r}\,'(b).\vec{r}\,'(b) - \vec{r}\,'(a).\vec{r}\,'(a)\right] \rightarrow \text{Fundamental Thm of Calculus.}$$

$$= \frac{1}{2}m\left[|\vec{r}\,'(b)|^2 - |\vec{r}\,'(a)|^2\right]$$

b) $W = \frac{1}{2}m|\vec{r}\,'(b)|^2 - \frac{1}{2}m|\vec{r}\,'(a)|^2 = K(B) - K(A)$.

c) $W = \int_C \vec{F}.d\vec{r} = \int_C -\nabla P.d\vec{r} \overset{\text{F.T of line integrals.}}{=} -(P(\vec{r}(b)) - P(\vec{r}(a)))$

$= -P(B) + P(A)$

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11) a) $x \geq 0$ and $4x^2 - 4y^2 - z^2 = 4$

$$\Rightarrow 4x^2 = 4 + 4y^2 + z^2 \quad \Rightarrow x = \sqrt{1 + y^2 + z^2/4}$$

Then the parametric equation is given by

$$y(u,v) = u, \quad z(u,v) = v, \quad x(u,v) = \sqrt{1 + u^2 + \frac{v^2}{4}} \, .$$

b) $x = x, \ y = y, \ z = x + 3, \ 0 \leq x^2 + y^2 \leq 1$

Then, $x = r\cos\theta$

$y = r\sin\theta$

$z = r\cos\theta + 3, \quad 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi$

12) $\vec{r}(u,v) = (1 - u^2 - v^2)\hat{i} - v\hat{j} - u\hat{k}$

$\vec{r}_u = -2u\hat{i} - \hat{k}$

$\vec{r}_v = -2v\hat{i} - \hat{j}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2u & 0 & -1 \\ -2v & -1 & 0 \end{vmatrix} = -1\hat{i} - (-2v)\hat{j} + 2u\hat{k}$$

The point $(-1,-1,-1)$ corresponds to $u = 1, v = 1$ and then

$$\vec{r}_u \times \vec{r}_v (1,1) = -\hat{i} + 2\hat{j} + 2\hat{k}$$

Then the equation of the tangent plane at $(-1,-1,-1)$ is

$$-1(x+1) + 2(y+1) + 2(z+1) = 0 \quad \Rightarrow -x + 2y + 2z = -3$$